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Representation Systems, Orthoposets and Quantum Logic

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We present a new approach of quantum logic and quantum systems description based on representation systems. This general algebraic formalism permits to represent systems from different points of view and reason about partial descriptions of it even though the descriptions are not available simultaneously (that is they can be associated to different points of view). We use a special form of these structures to define a method for decomposing orthoposets into boolean algebras, in such a way that it permits to consider more properties about a quantum system than using usual methods, and to define general (and, we hope, intuitive) embeddings of quantum structures into Heyting algebras.

KEY WORDS: quantum logic; intuitionistic logic; knowledge representation.

1. INTRODUCTION

The logical study of quantum mechanics, originated in the thirties by Birkhoff and von Neumann (1936), aims at investigating formally what makes quantum mechanics so different from the classical world. It is based on the use of *closed subspaces* of Hilbert spaces for representing properties about a quantum system, and on similarities between usual operations on subspaces and logical connectives. Since then, many attempts have been made to identify which properties of Hilbert spaces where responsible for the rupture between quantum mechanics and classical newtonian mechanics (Mackey, 1957; Piron, 1976; Pták and Pulmannová, 1991) and have led to many forms of quantom logics, the most standard one being based on orthomodular lattices (Hughes, 1989; Svozil, 1998; Chiara and Giuntini, 2001).

Unfortunately, despite the large amount of publications available on this topic, these works have remained extremely theoretical, have led to very little applications and are still subject to severe criticisms (see for instance Girard, 2003).

In this article, we present an approach of quantum logic and quantum systems description based on representation systems (Brunet, 2002a,b, 2003a,b). This

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general algebraic formalism permits to represent systems from different points of view and reason about partial descriptions of it even though the descriptions are not available simultaneously (that is they can be associated to different points of view). With the restriction that for every point of view, the descriptions associated to it form a boolean algebra (which structure is usually seen as an emblematic representation of classical information), every representation systems can be associated to an orthoposet, and conversely, every orthoposet can be obtained from a boolean representation system.

The relations between these two kinds of structures, and some properties of boolean representation systems, permit to develop a general way to extend orthoposets by adding partial descriptions, which di not correspond to any point of view, but which are relevant, with regards to the considered boolean representation system. This completion method enables to manipulate more expressive descriptions of the system. Moreover, using a particular type of completions, we show that it is possible to embed an orthoposet into a Heyting algebra in such a way that the partial order and the orthocomplement are preserved. Thus, considering our approach where elements of the original orthoposet are seen as partial descriptions of a system, this constitutes a general (and, we think, intuitive) way to obtain a logical way to study it.

The article is divided as follows: we first introduce representation systems in section 2 and then restrict to boolean representation systems and study the relationships that exist between these structures and orthoposets (section 3). Then, we use a decomposition of an orthoposet into boolean algebras using boolean representation systems to define a notion of consistency on elements of an orthoposet $\mathcal{P}$. This permits to extend it by considering collections of elements of $\mathcal{P}$ which behave in a convenient way (section 4). Finally, in section 5, we show how these constructions permit to define a general way to embed orthoposets into Heyting algebras.

2. REPRESENTATION SYSTEMS

The starting idea for representing partial descriptions of a system is to use a partially order set (or *poset* for short) $\langle \mathcal{P}, \leq \rangle$ where elements of $\mathcal{P}$ stand for partial descriptions of a given situation and $\leq$ is a partial order defined on $\mathcal{P}$, so that if two elements $x, y \in \mathcal{P}$ are such that $x \leq y$, then x corresponds to more information than y.

Example 2.1. (Firefly on a Box) Consider a firefly trapped inside a closed box divided into four quadrants by transparent walls (depicted in the Fig. 1(a). If one observes the box from below, two partial descriptions can be obtained: either the firefly is on the *left* side, or it is on the *right* side. By adding a third partial description yielding no information at all (which we will call *top* and write $\top$), we get a three-elements poset: $\mathcal{P}_x = \{\top, \textit{left}, \textit{right}\}$. We define the partial order $\leq$ by

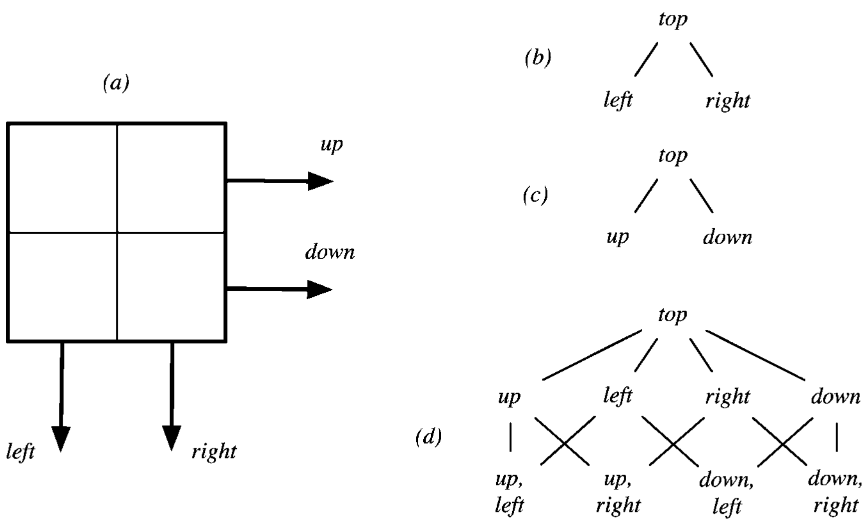

Fig. 1. Firefly in a box.

stating that $\top$ is the greatest element and the other two elements are not comparable. This poset is depicted in Fig. 1(b), together with $\mathcal{P}_y$(c), based on informations such as *up* and *down*, and their product $\mathcal{P}_x \times \mathcal{P}_y$ (d).

Representation systems (Brunet, 2002a,b, 2003a,b) are general algebraic structures intended to model partial descriptions of a system. They are based on a collection of posets, each corresponding to a way to observe the system, to a point of view. Thus, given a set $\mathcal{I}$ indexing the different points of view, we consider a collection of posets $\{\mathcal{P}_i | i \in \mathcal{I}\}$, so that for each $i \in \mathcal{I}$, $\mathcal{P}_i$ is a set of partial descriptions related to point of view i.

Now, since all these posets are seen as partial representations of a single system, we introduce a collection of functions $\{f_{i|j} : \mathcal{P}_j \to \mathcal{P}_i | i, j \in \mathcal{I}\}$ called *transformation functions* which purpose is, given a partial description $x \in \mathcal{P}_i$, to transform it into a description $f_{j|i}(x) \in \mathcal{P}_j$. The intuition is that starting from an element $x \in \mathcal{P}_i$, during the translation process, no information is added and as little information as possible is lost. We formalize these functions using three properties:

Identity	$\forall i \in \mathcal{I}, \forall x \in \mathcal{P}_i, f_{i|i}(x) = x$
Monotony	$\forall i, j \in \mathcal{I}, \forall x, y \in \mathcal{P}_i, x \leq_i y \Rightarrow f_{j|i}(x) \leq_j f_{j|i}(y)$
Composition	$\forall i, j, k \in \mathcal{I}, \forall x \in \mathcal{P}_i, f_{k|i}(x) \leq_k f_{k|j} \circ f_{j|i}(x)$

corresponds to the fact that nothing has to be done for translating an element of $\mathcal{P}_i$ to itself. *Monotony* expresses the fact that no information is added: if a description $x \in \mathcal{P}_i$ is more informative than y (that is $x \leq_i y$), then every information in $f_{j|i}(y)$

is both present in y and expressible in $\mathcal{P}_j$, so that it also has to be present in $f_{j|i}(x)$. Finally, the meaning of *Composition* is that information may be lost in the process: for $i, j \in \mathcal{I}$ and $x \in \mathcal{P}_i$, x should be seen as more informative than $f_{j|i}(x)$. But since these two elements to not belong to the same poset, they cannot be compared directly. Instead, we demand that translated into every poset $\mathcal{P}_k$, one gets a more informative element if starting from x rather than from $f_{j|i}(x)$. In other words, for all $k \in \mathcal{I}$, one has $f_{k|i}(x) \leq_k f_{k|j} \circ f_{j|i}(x)$. We summarize this as follows:

Definition 2.1. (Representation System) A representation system is a triplet

$$\mathcal{S} = \langle \mathcal{I}, \{\mathcal{P}_i\}_{i \in \mathcal{I}}, \{f_{i|j}\}_{i,j \in \mathcal{I}} \rangle$$

where every $\mathcal{P}_i$ is a poset, and where the transformation functions $\{f_{i|j} : \mathcal{P}_j \rightarrow \mathcal{P}_i\}$ verify the *identity*, *monotony* and *composition* properties introduced above.

Example 2.2. (Firefly in a box) The previous example can be turned into a representation system, by considering $\mathcal{I} = \{x, y\}$. The posets $\mathcal{P}_x$ and $\mathcal{P}_y$ are the same as those defined earlier. Concerning the transformation functions, since knowing for instance that the firefly is on the left- or right-side of the box does not tell anything whether it is up or down, it follows that $\forall u \in P_x$, $f_{y|x}(u) = \top_y$. Similarly, $f_{x|y}$ maps elements of $\mathcal{P}_y$ to $\top_x$. By adding $f_{x|x}$ and $f_{y|y}$ which are the identity functions, it is easy to verify that the obtained structure is a representation system.

Example 2.3. (Time of earth) Consider the Earth at a given moment, and define points of view as corresponding to time zones, characterized by their difference with the G.M.T. zone. For every time zone, the possible partial description have the form of an interval $[b, e]$ where b, e are integers in the set $\{0, 1, 2, \ldots, 23\}$. They correspond to assertions of the form "the local time is between b and e" and are partially ordered by the overlapping relation, modulo 24. For instance, $[3, 4] \leq [2, 7] \leq [23, 8]$.

To turn an interval from a time zone t_1 to a time zone t_2, the natural definition is $f_{t_2|t_1}([b, e]) = [\lfloor b - t_1 + t_2 \rfloor, \lceil e - t_1 + t_2 \rceil]$ where addition is considered modulo 24. For instance, if the time is between 12 and 14 in the GMT time zone ($t_1 = 0h$), it is between 4 and 6 Pacific time ($t_2 = -8h$): $f_{-8h|0h}([12, 14]) = [4, 6]$. One can easily show that these definitions do correspond to a representation system. Moreover, considering time zones which have a non integer difference with G.M.T. (such as India, $+5h30$) illustrates the role of the *composition* property: $[4, 6] = f_{-8h|0h}([12, 14]) \leq f_{-8h|+5h30} \circ f_{+5h30|0h}([12, 14]) = [3, 7]$.

A last example shows how transformation systems do closely relate to some algebraic structures and functions such as Galois connections and closure operators (Erné *et al.*, 1992; Birkhoff, 1967; Gratzer, 1978).

Definition 2.4. (Upper closure operator) Given a poset $\mathcal{P}$, an *upper closure operator* $\rho : \mathcal{P} \to \mathcal{P}$ is a monotonous function which verifies:

$$\forall x, x \leq \rho(x) \quad \forall x, \rho(x) = \rho \circ \rho(x) \tag{1}$$

An upper closure operator ρ defined on a poset $\mathcal{P}$ can conveniently be used to represent an approximation operation: given an element x of $\mathcal{P}$ seen as a partial description, applying ρ corresponds to losing some information (so that $x \leq \rho(x)$ or, in other words, x is more precise than $\rho(x)$), and since $\rho \circ \rho(x) = \rho(x)$, no more information can be lost by applying ρ again. We now show that a representation system can be obtained from a collection of upper closure operators defined on a single poset.

Example 2.5. (Poset approximation) Consider a poset $\mathcal{P}$ and a collection $\{\rho_i\}_{i \in \mathcal{I}}$ of upper closure operators on $\mathcal{P}$ and define for every $i \in \mathcal{I}$, $\mathcal{P}_i = \rho_i(\mathcal{P}) = \{x \in \mathcal{P} | x = \rho_i(x)\}$ and for $i, j \in \mathcal{I}$, $f_{i|j}$ as the restriction of ρ_i to $\mathcal{P}_j$. This way, $\langle \mathcal{I}, \{\mathcal{P}_i\}, \{f_{i|j}\} \rangle$ is a representation system.

More details about this relationship can be found in (Brunet, 2002a, 2003, in press), and we only mention that the three properties characterizing transformation functions (that is *Identity*, *Monotony* and *Composition*) can actually be derived from the definition of Galois connections.

3. RESTRICTION TO BOOLEAN ALGEBRAS

We will now restrict ourselves to a special class of representation systems, where each poset is a boolean algebra. This way, every partial description of the system can be seen as classical information, since in classical systems, observable events do form a boolean algebra. A few adaptations have to be applied to our formalism, in order to have our transformation functions behave properly with regards to the additional connectives: we will add two properties, one for negation and another one for disjunction.

Suppose first that one has $x \in \mathcal{P}_i$ and $y \in \mathcal{P}_j$ such that $f_{j|i}(x) \leq y$. This means that x, even after some loss of information due to the translation process, is more informative than y. We want to use this information to compare $x^\perp$ and $y^\perp$, since intuitively, $y^\perp$ would in that case be more informative than $x^\perp$, which can be expressed as $f_{i|j}(y^\perp) \leq_i x^\perp$. Thus, we define a new property:

Inversion $\forall i, j \in \mathcal{I}, \forall x \in \mathcal{P}_i, \forall y \in \mathcal{P}_j, f_{j|i}(x) \leq_j y \Rightarrow f_{i|j}(y^\perp) \leq_i x^\perp$

Now, we will add another condition concerning the disjunction to represent the fact that in the application of a transformation function, one loses as little information as possible. This means that if an element $z \in \mathcal{P}_j$ is more general that both $x \in \mathcal{P}_i$ and $y \in \mathcal{P}_i$ (that is $f_{j|i}(x) \leq_j z$ and $f_{j|i}(y) \leq_j z$), then it is more general than $x \vee y$. From this follows the equality between $f_{j|i}(x \vee y)$ and $f_{j|i}(x) \vee f_{j|i}(y)$ by taking $z = f_{j|i}(x) \vee f_{j|i}(y)$. It is this equality which will use as our last property:

Join-Preservation $\forall i, j \in \mathcal{I}, \forall x, y \in \mathcal{P}_i, f_{j|i}(x \vee y) = f_{j|i}(x) \vee f_{j|i}(y)$

It should be noted that an equivalent property cannot be defined for conjunction, due to the possible loss of information. For instance, it might happen that two elements x and y are such that $f_{j|i}(x) = f_{j|i}(y) = \top_j$ while $f_{j|i}(x \wedge y) < \top_j$. We obtain the following definition:

Definition 3.1. (Boolean representation system) A *boolean representation system* (or b.r.s. for short) is a representation system $\mathcal{S}$ of the form $\langle \mathcal{I}, \{\mathcal{B}_i\}, \{f_{i|j}\}\rangle$ where every $\mathcal{B}_i$ is a boolean algebra and such that the transformation functions verify the *inversion* and *join-preservation* properties.

It should be noted that in a boolean representation system, extremal elements are mapped to extremal elements:

Proposition 3.1. *Given a b.r.s.* $\mathcal{S} = \langle \mathcal{I}, \{\mathcal{B}_i\}_{i\in\mathcal{I}}, \{f_{i|j}\}_{i,j\in\mathcal{I}}\rangle$, *one has for all* $i, j \in \mathcal{I}$:

$$f_{j|i}(\bot_i) = \bot_j \quad f_{j|i}(\top_i) = \top_j \tag{2}$$

Proof: From the *inversion* property, since $f_{i|j}(\top_j) \leq \top_i$, one has $f_{j|i}(\bot_i) \leq \bot_j$ from which follows the equality. For the greatest elements, by definition of $\top_i$, one has $f_{i|j}(f_{j|i}^{\perp}(\top_i)) \leq_i \top_i$. Now, using the *monotony* and *composition* properties of transformation functions, it follows that:

$$f_{j|i}^{\perp}(\top_i) \leq_j f_{j|i} \circ f_{i|j}\left(f_{j|i}^{\perp}(\top_i)\right) \leq_j f_{j|i}(\top_i) \tag{3}$$

One concludes by remark that $\top$ is the only element of a boolean algebra such that $x^{\perp} \leq x$. □

3.1. Order Considerations

Given a boolean representation system $\mathcal{S} = \langle \mathcal{I}, \{\mathcal{B}_i\}, \{f_{i|j}\}\rangle$, it is possible to extend the order relation of each boolean algebra to a global partial order. Following the intuition behind the definition of representation systems, given an element $x \in \mathcal{B}_i$, its image $f_{j|i}(x)$ in $\mathcal{B}_j$ is supposed to correspond to fewer information than x, since some information might have been lost in the transformation, and no other information have been added. More generally, for an element $y \in \mathcal{B}_j$, if $f_{j|i}(x) \leq_j y$, we will think of x as being more informative than y. We will use this relation to compare all partial description of a representation system:

Definition 3.2. Using the previous notations, let $\mathcal{S}_\star = \{\langle i, x\rangle | i \in \mathcal{I} \, and \, x \in \mathcal{B}_i\}$ be the disjoint union of $\mathcal{S}$'s boolean algebras, and define the binary relation $\leq_\star$ on $\mathcal{S}_\star$ by:

$$\langle i, x\rangle \leq_\star \langle j, y\rangle \quad \Leftrightarrow \quad f_{j|i}(x) \leq_j y \tag{4}$$

Proposition 3.2. *The binary relation $\leq_\star$ is a pre-order, that is it is reflexive and transitive.*

Proof: Reflexivity corresponds to the *identity* property. Transitivity is a direct consequence of the *composition* and *monotony* properties of transformation functions: if $f_{j|i}(x) \leq_j y$ and $f_{k|j}(y) \leq_k z$, then $f_{k|i}(x) \leq f_{k|j} \circ f_{j|i}(x) \leq_k f_{k|j}(y) \leq_k z$. □

This pre-order has very strong relationships with the partial orders $\{\leq_i\}$ of the different boolean algebras of $\mathcal{S}$.

Proposition 3.3. *The relation $\leq_\star$ extends the partial order of each boolean algebra of $\mathcal{S}$:*

$$\forall i \in \mathcal{I}, \forall x, y \in \mathcal{B}_i, x \leq_i y \Leftrightarrow \langle i, x\rangle \leq_\star \langle i, y\rangle \tag{5}$$

Proof: It is a direct consequence of the *identity* property: $\langle i, x\rangle \leq_\star \langle i, y\rangle \Leftrightarrow f_{i|i}(x) \leq_i y \Leftrightarrow x \leq_i y$. □

The set $\mathcal{S}_\star$ can be equipped with an orthocomplementation operation $\cdot^{\perp_\star}$, mapping an element $\langle i, x\rangle$ to $\langle i, x\rangle^{\perp_\star} = \langle i, x^\perp\rangle$.

Proposition 3.4. *For all x and y in $\mathcal{S}_\star$, one has:*

$$x \leq_\star y \Rightarrow y^{\perp_\star} \leq_\star x^{\perp_\star} \quad x^{\perp_\star \perp_\star} = x \tag{6}$$

Proof: The implication is a direct consequence of the definitions of $\cdot^{\perp_\star}$ and the *inversion* property of transformation functions: $\langle i, x\rangle \leq_\star \langle j, y\rangle \Rightarrow f_{j|i}(x) \leq_j y \Rightarrow f_{i|j}(y^\perp) \leq_i x^\perp \Rightarrow \langle j, y^\perp\rangle \leq_\star \langle i, x^\perp\rangle$. □

3.2. Obtention of a Poset

Since the relation $\leq_\star$ defined so far is just a pre-order and not a partial order, we will define the equivalence relation $\simeq_\star$ associated to $\leq_\star$ in order to turn $\mathcal{S}_\star$ into a poset.

Definition 3.3. Let the relation $\simeq_\star$ be defined on $\mathcal{S}_\star \times \mathcal{S}_\star$ by:

$$\langle i, x\rangle \simeq_\star \langle j, y\rangle \quad \Leftrightarrow \quad \langle i, x\rangle \leq_\star \langle j, y\rangle \textit{ and } \langle j, y\rangle \leq_\star \langle i, x\rangle \tag{7}$$

It is obviously an equivalence relation on $\mathcal{S}_\star$, permitting to identify those elements of $\mathcal{S}_\star$ which cannot be distinguished with regards to $\leq_\star$. Let us introduce a few notations: first, given an element $x \in \mathcal{S}_\star$, $[x]$ will denote the equivalence class of x, i.e. the set $\{y|x \simeq_\star y\}$. $\mathcal{S}_\simeq$ will denote the quotient of $\mathcal{S}_\star$ by $\simeq_\star$, so that its elements are the equivalence classes of $\mathcal{S}_\star$, and $\leq_\simeq$ is the pre-order relation restricted to the equivalence classes. It should be remark that this relation is well defined, and one actually has:

Proposition 3.5. *The pair $\langle \mathcal{S}_\simeq \leq_\simeq\rangle$ is a poset.*

Proof: It should be first noted that for all $\langle i, x\rangle$ and $\langle j, y\rangle$ in $\mathcal{S}_\star$, one has:

$$\langle i, x\rangle \leq_\star \langle j, y\rangle \Leftrightarrow [\langle i, x\rangle] \leq_\simeq [\langle j, y\rangle] \tag{8}$$

This permits to show the reflexivity and transitivity of $\leq_\simeq$. Now, for the anti-symmetry, suppose that both $[\langle i, x\rangle] \leq_\simeq [\langle j, y\rangle]$ and $[\langle j, y\rangle] \leq_\simeq [\langle i, x\rangle]$. This implies that $\langle i, x\rangle \simeq_\star \langle j, y\rangle$, so that $[\langle i, x\rangle] = [\langle j, y\rangle]$. □

Proposition 3.6. *For all $\langle i, x\rangle$ and $\langle j, y\rangle$ in $\mathcal{S}_\star$, one has:*

$$\langle i, x\rangle \simeq_\star \langle j, y\rangle \Leftrightarrow (f_{j|i}(x) = y \text{ and } f_{i|j}(y) = x) \tag{9}$$

Proof: The $\Leftarrow$-direction is a direct consequence of the definition of $\simeq_\star$. Conversely, one has if $\langle i, x\rangle \simeq_\star \langle j, y\rangle$, then $f_{j|i}(x) \leq_j y$ and $f_{i|j}(y) \leq_i x$. Applying $f_{j|i}$ to the second inequality, one has $y \leq f_{j|i} \circ f_{i|j}(y) \leq_j f_{j|i}(x)$, so that $y = f_{j|i}(x)$, and similarly, $x = f_{i|j}(y)$. □

The same way as we have defined a complement operation for $\mathcal{S}_\star$, we can define operation $\cdot^{\perp_\simeq}$ on $\mathcal{S}_\simeq$ by: $[\langle i, x^{\perp_\simeq}\rangle] = [\langle i, x^{\perp_\star}\rangle] = [\langle i, x^{\perp}\rangle]$. Proposition 3.4 ensures that this operation is well defined on equivalence classes, and one has:

Proposition 3.7. *The tuple $\langle \mathcal{S}_\simeq, \leq_\simeq, \cdot^{\perp_\simeq}\rangle$ is an orthoposet.*

Proof: This is a direct consequence of proposition 3.4:

$$[x] \leq_\simeq [y] \Rightarrow x \leq_\star y \Rightarrow y^{\perp_\star} \leq_\star x^{\perp_\star} \Rightarrow [y]^{\perp_\simeq} \leq_\simeq [x]^{\perp_\simeq} \tag{10}$$

Likewise, one has $[x]^{\perp_\simeq \perp_\simeq} = [x^{\perp_\star \perp_\star}] = [x]$. □

We finish our characterization of the structure of $\mathcal{S}_\star$ by showing that it has extremal elements, and that partial join and meet operations can be defined.

Proposition 3.8. *For all $i \in \mathcal{I}$ and $\langle j, x\rangle \in \mathcal{S}_\star$, one has: $\langle j, x\rangle \leq_\star \langle i, \top\rangle$ and $\langle i, \bot\rangle \leq_\star \langle j, x\rangle$.*

Proof: This follows from the definition of $\top$ and $\bot$. □

This shows that for all $i, j \in \mathcal{I}$, $[\langle i, \bot\rangle] = [\langle j, \bot\rangle]$ and $[\langle i, \top\rangle] = [\langle j, \top\rangle]$ and if $\bot_\simeq$ (resp. $\top_\simeq$) denotes $[\langle i, \bot\rangle]$ (resp. $[\langle i, \top\rangle]$) for some i in $\mathcal{I}$, then $\bot_\simeq$ (resp. $\top_\simeq$) is the least (resp. greatest) element of $\mathcal{S}_\simeq$.

Proposition 3.9. *Given an indice $i \in \mathcal{I}$ and two elements $x, y \in \mathcal{B}_i$, the meet (resp. join) of $[\langle i, x\rangle]$ and $[\langle i, y\rangle]$ exists and is equal to $[\langle i, x \vee y\rangle]$(resp. $[\langle i, x \wedge y\rangle]$).*

Proof: First, from $x \leq_i x \vee y$, it follows that $[\langle i, x\rangle] \leq_\star [\langle i, x \vee y\rangle]$ and similarly that $[\langle i, y\rangle] \leq_\star [\langle i, x \vee y\rangle]$. Now, let $\langle j, z\rangle$ be in $\mathcal{S}_\star$ and suppose that $\langle i, x\rangle \leq_\star \langle j, z\rangle$ and $\langle i, y\rangle \leq_\star \langle j, z\rangle$. One then has $f_{j|i}(x \vee y) = f_{j|i}(x) \vee f_{j|i}(y) \leq_j z$ from the *join-preservation* property. Rewritten in terms of $\mathcal{S}_\simeq$, we have shown that:

$$\forall z \in \mathcal{S}_\simeq, ([\langle i, x\rangle] \leq_\simeq z \quad \text{and} \quad [\langle i, y\rangle] \leq_\simeq z) \Leftrightarrow [\langle i, x \vee y\rangle] \leq_\simeq z \tag{11}$$

Thus, the join of $[\langle i, x\rangle]$ and $[\langle i, y\rangle]$ (which we write $[\langle i, x\rangle] \vee_\simeq [\langle i, y\rangle]$) exists and equals $[\langle i, x \vee y\rangle]$. The proof for the meet is even simpler: if $f_{i|j}(z) \leq_i x$ and $f_{i|j}(z) \leq_i y$ then $f_{i|j}(z) \leq_i x \wedge y$. □

Corollary 3.1. *For every element x in $\mathcal{S}_\simeq$, $x \vee_\simeq x^{\bot_\simeq}$ (resp. $x \wedge_\simeq x^{\bot_\simeq}$) exists and is equal to $\top_\simeq$ (resp. $\bot_\simeq$).*

All these results are summarized in the following theorem:

Theorem 3.1. *Given a pointed boolean representation system $\mathcal{S}$, and using the previous notations, the tuple* $\mathrm{OP}(\mathcal{S}) = \langle \mathcal{S}_\simeq, \leq_\simeq, \cdot^{\bot_\simeq}, \bot_\simeq, \top_\simeq\rangle$ *is a bounded orthoposet.*

For the sake of simplicity, from now on, the term orthoposet will always correspond to a bounded orthoposet.

3.3. Projective Boolean Subalgebras

Given a b.r.s. $\mathcal{S}$, the boolean algebras in $\mathcal{S}$ are special boolean subalgebras of $\mathrm{OP}(\mathcal{S})$, which can be given a simple characterization.

Definition 3.4. Given an orthoposet $\mathcal{P}$, a boolean subalgebra $\mathcal{B}$ of $\mathcal{P}$ is said to be *projective* if and only if every element x of $\mathcal{P}$ has a projection $x_\mathcal{B}$ in $\mathcal{B}$, that is an element such that: $x \leq x_\mathcal{B}$ and $\forall y \in \mathcal{B}, x \leq y \Rightarrow x_\mathcal{B} \leq y$.

Equivalenty, a boolean subalgebra $\mathcal{B}$ is projective if and only if there exists an upper closure operator $\rho_{\mathcal{B}}$ such that $x \in \mathcal{B} \Leftrightarrow x = \rho_{\mathcal{B}}(x)$.

Proposition 3.10. *Every complete boolean subalgebra $\mathcal{B}$ of an orthoposet $\mathcal{P}$ is projective.*

Proof: This comes from the fact that for every element x of $\mathcal{P}$, $x_{\mathcal{B}}$ can be defined as $\bigwedge\{y \in \mathcal{B} | x \leq y\}$. □

Proposition 3.11. *Given a b.r.s. $\mathcal{S}$, with the usual notations, for all $i \in \mathcal{I}$, $\mathcal{B}_i$ is a projective boolean subalgebra of* OP($\mathcal{S}$).

Proof: For $x \in \mathrm{OP}(\mathcal{S})$, there exists an indice $i \in \mathcal{I}$ and an element $x_i \in \mathcal{B}_i$ such that $x = [\langle i, x_i\rangle]$. Then for every $j \in \mathcal{I}$, the projection of x on $\mathcal{B}_j$ is given by $[\langle j, f_{j|i}(x_i)\rangle]$. □

The notion of projective boolean subalgebras of an orthoposet is closely related to that of boolean representation system:

Proposition 3.12. *Let $\mathcal{P}$ be an orthoposet and $\{\mathcal{B}_i\}_{\mathcal{I}}$ a collection of projective boolean subalgebras of $\mathcal{P}$ such that $\mathcal{P} = \bigcup_i \mathcal{B}_i$. For $i, j \in \mathcal{I}$, define $f_{i|j}$ as the function mapping every element x of $\mathcal{B}_j$ to its projection $x_{\mathcal{B}_i}$ on $\mathcal{B}_i$. Then $\mathcal{S} = \langle \mathcal{I}, \{\mathcal{B}_i\}, \{f_{i|j}\}\rangle$ is a boolean representation system and $\mathcal{P} =$* OP($\mathcal{S}$).

Proof: To prove that $\mathcal{S}$ is a b.r.s., one only needs to show that $\{f_{i|j}\}$ is a collection of transformation functions, which follows directly from their definition. Now, for every elements $\langle i, x\rangle$ and $\langle j, y\rangle$ in $\mathcal{S}_\star$, one has $\langle i, x\rangle \leq_\star \langle j, y\rangle \Leftrightarrow f_{j|i}(x) \leq_j y \Leftrightarrow x \leq_{\mathcal{P}} y$. This implies the equality $\mathcal{P} = \mathrm{OP}(\mathcal{S})$. □

Thus, given an orthoposet $\mathcal{P}$, a boolean representation system $\mathcal{S}$ such that $\mathrm{OP}(\mathcal{S}) = \mathcal{P}$ can also be defined as a collection $\{\mathcal{B}_i\}$ of projective boolean subalgebras of $\mathcal{P}$ verifying $\mathcal{P} = \bigcup \mathcal{B}_i$. This equivalent way to describe boolean description systems permits to easily prove that every orthoposet can be obtained from a boolean representation system.

Theorem 3.2. (Representation of Orthoposets). *For every orthoposet $\mathcal{P}$, there exists a boolean representation system $\mathcal{S}$ such that $\mathcal{P} \simeq$* OP($\mathcal{S}$).

Proof: For every element $x \in \mathcal{P} \setminus \{\top, \bot\}$, the subset $\mathcal{B}_x = \{\top, \bot, x, x^\perp\}$ is a boolean subalgebra of $\mathcal{P}$. Moreover, since it is finite, it is a complete boolean algebra, so that from proposition 1, it is a projective boolean subalgebra of $\mathcal{P}$.

Now, define $\mathbf{2}^{\mathcal{P}}$ as the b.r.s. corresponding to the collection $\{\mathcal{B}_x | x \in \mathcal{P} \setminus \{\top, \bot\}\}$. From proposition 3.12 follows the expect result: $\mathcal{P} = \mathrm{OP}(\mathbf{2}^{\mathcal{P}})$. □

Given an orthoposet, BRS($\mathcal{P}$) will denote the set of boolean representation systems $\mathcal{S}$ such that $\mathrm{OP}(\mathcal{S}) \simeq \mathcal{P}$. In particular, $\mathbf{2}^{\mathcal{P}} \in \mathrm{BRS}(\mathcal{P})$.

These two theorems are extremely interesting in the light of quantum logic, since they give a characterization of orthoposets in terms of boolean representation families, which we have defined as a collection of classical points of view of a given system. Indeed, orthoposets are algebraic structures which are very close to orthoalgebras and orthomodular posets (Foulis *et al.*, 1992; Foulis and Randall, 1981; Wilce, 2000) which play a central role in quantum logic (Chiara and Giuntini, 2001; Hughes, 1989).

The expression of this type of structure in terms of a decomposition as boolean algebras already exists in many ways in the litterature (one can mention pastings Dichtl, 1981; Greechie, 1971; Navara, 2000, semi-pastings Navara, 2000, partial boolean algebras Hughes, 1985, for instance), and appears to be a powerful for the study of orthomodular structures.

The decomposition which we have introduced above has the particularity to stress the fact that the different boolean algebras used in the decomposition correspond to different points of view of a system.

4. CONSISTENCY

We introduce a notion of consistency among the elements of a representation system (not necessarily boolean) and use this notion to construct a completion of the representation system, where one considers collections of consistent partial descriptions. This way, one gets a poset which elements can be envisioned as descriptions of the state of the system corresponding to the initial representation system, and are usually more informative than the original partial descriptions considered on their own.

4.1. Definitions

It might happen that two descriptions, belonging to different posets, do correspond to a single situation. This happens when each description provides more information in its own poset that the other. Formally, this corresponds to the following situation:

Definition 4.1 (Co-consistency). Given two indices $i, j \in \mathcal{I}$, an element $x \in \mathcal{P}_i$ and another element $y \in \mathcal{P}_j$, the pairs $\langle i, x\rangle$ and $\langle j, y\rangle$ are *co-consistent* if and only if they verify:

$$x \leq_i f_{i|j}(y) \quad y \leq_j f_{j|i}(x) \tag{12}$$

Example 4.1. In the example of the firefly in a box, the two descriptions $\langle x, \textit{left}\rangle$ and $\langle y, up\rangle$ are co-consistent.

In the time zones example, the two descriptions $\langle 0h, [12, 13]\rangle$ and $\langle 5h30, [17, 18]\rangle$ are co-consistent. In that case, it means that the time in the G.M.T. zone is between $12h00$ and $12h30$, even though the latter information cannot be represented exactly.

This illustrates the fact that a pair of co-consistent elements provides strictly more information about the system. We extend this notion by considering consistent families, containing an element for each poset of the representation system:

Definition 4.2 (Consistent Family). A consistent family is an element $\mathbf{x} = \{x_i\}$ in $\prod_{i\in\mathcal{I}} \mathcal{P}_i$ such that:

$$\forall i, j \in \mathcal{I}, x_i \leq_i f_{i|j}(x_j) \tag{13}$$

Following the intuition provided by pairs of co-consistent descriptions, a consistent family can be envisioned as a partial description of the system which is more precise than any of its components, but which cannot in general be related to a single point of view. One might wonder whether there exists consistent families for a given representation system, and we now show that it is always the case, and that every element of every poset can be used to define a consistent family:

Proposition 4.1. *Given an indice $i \in \mathcal{I}$ and an element $u \in \mathcal{P}_i$, the family $\{f_{j|i}(u)\}_{j\in\mathcal{I}}$ is consistent.*

Proof: This results from the *composition* property: $f_{k|i}(u) \leq f_{k|j}(f_{j|i}(u))$. □

Moreover, the set of consistent families can be turned into a poset, using the pointwise partial order and in that case, every poset of the original representation system can be obtained as an approximation as in example *Poset Approximation* p. 8:

Proposition 4.2. *Given a representation system $\mathcal{S} = \langle \mathcal{I}, \{\mathcal{P}\}_i, \{f_{i|j}\}\rangle$, define the poset $\mathcal{P}_\mathcal{S}$ as the collection of consistent families of $\mathcal{S}$, ordered pointwise, and for every $i \in \mathcal{I}$, define ρ_i on $\mathcal{P}_\mathcal{S}$ as $\rho_i(\{x_j\}_j) = \{f_{k|i}(x_i)\}_k$. Then:*

1. *For all $i \in \mathcal{I}$, ρ_i is an upper closure operator on $\mathcal{P}_\mathcal{S}$.*
2. *For all $i \in \mathcal{I}$, $\mathcal{P}_i$ is isomorphic to $\{\mathbf{x} \in \mathcal{P}_i | \mathbf{x} = \rho_i(\mathbf{x})\}$.*
3. *For all $i, j \in \mathcal{I}$, $f_{i|j} = \rho_i|_{\mathcal{P}_j}$ with the previous isomorphism.*

Proof: For the first point, the monotony of transformation functions permits to show both the monotony and the fact that $\forall x, x \leq \rho_i(x)$. Since $f_{i|i}$ is the identity on $\mathcal{P}_i$, it follows that $\rho_i \circ \rho_i = \rho_i$.

For the second point, a consistent family $\mathbf{x}$ verifying $\mathbf{x} = \rho_i(\mathbf{x})$ is such that $\forall j, x_j = f_{j|i}(x_i)$, so that the pair of functions $\mathbf{x} \mapsto x_i$ and $x \mapsto \{f_{j|i}(x)\}_j$ forms an isomorphism between $\{\mathbf{x} \in \mathcal{P}_i | \mathbf{x} = \rho_i(\mathbf{x})\}$ and $\mathcal{P}_i$. The last point is a direct consequence of the second one. □

Let us now turn back to boolean representation systems, and study the way consistent families of a b.r.s. $\mathcal{S}$ relate to OP($\mathcal{S}$).

4.2. Consistent Representations and Completions

Let $\mathcal{S}$ be a b.r.s. and, using the usual notations, let x be an element of $\mathcal{B}_i$. From proposition 4.1, x can be used to define a consistent family $\{f_{j|i}(x)\}_j$. For all $j \in \mathcal{I}$, one has $\langle i, x\rangle \leq_\star \langle j, f_{j|i}(x)\rangle$ so that for this consistent family, only $x = f_{i|i}(x)$ appears to be the only relevant element, if one considers information. This suggests that only least elements of a consistant family are important, so that they may fruitfully be associated to filters of OP($\mathcal{S}$). To develop this, let us first remind a few notations and introduce some notations.

Definition 4.3 (Filter). Given a poset $\mathcal{P}$, a *filter* of $\mathcal{P}$ is a subset $F \subseteq \mathcal{P}$ such that for all $x \in I$ and $y \in \mathcal{P}$, if $x \leq y$, then $y \in F$. In other words, a filter of a poset is an upwards-closed subset.

In the following, $\wp^\uparrow(\mathcal{P})$ will denote the set of filters of a poset $\mathcal{P}$, and given an element $x \in \mathcal{P}$, $x^\uparrow$ will correspond to the filter $\{y \in \mathcal{P} | x \leq y\}$. Such a filter is said to be *principal*.

Thus, as stated previous, we will associate consistant families of a b.r.s. $\mathcal{S}$ to filters of OP($\mathcal{S}$) and, more precisely, we will use consistent families to characterize classes of filters of a bounded orthoposet:

Definition 4.4 (Consistent Representation of Filters). Given a bounded orthoposet $\mathcal{P}$ and a b.r.s. $\mathcal{S} \in$ BRS($\mathcal{P}$), a filter $F \in \wp^\uparrow(\mathcal{P})$ has a consistent representation in $\mathcal{S}$ if and only if there exists a consistent family $\mathbf{x} = \{x_i\}$ in $\mathcal{S}$ such that:

$$F = \bigcup_{i \in \mathcal{I}} [\langle i, x_i\rangle]^\uparrow \tag{14}$$

It is easy to express the fact that a filter has a consistent representation in a given b.r.s. in terms of projective boolean subalgebras. First, for a filter F of an orthoposet $\mathcal{P}$ and for a projective boolean subalgebra $\mathcal{B}$ is $\mathcal{P}$, we will say that F is closed for $\mathcal{B}$ if and only if $F \cap \mathcal{B}$ is a principal filter of $\mathcal{B}$.

Proposition 4.3. *A filter $F \in \wp^{\uparrow}(\mathcal{P})$ has a consistent representation in $\mathcal{S}$ if and only if for all $i \in \mathcal{I}$, F is closed for $\mathcal{B}_i$.*

Proof: First, let $\mathbf{x} = \{x_i\}$ be a consistent family of $\mathcal{S}$ such that $F = \bigcup_i [\langle i, x_i\rangle]^{\uparrow}$. From the equality $F \cap \mathcal{B}_i = \bigcup_j ([\langle i, x_i\rangle]^{\uparrow} \cap \mathcal{B}_i)$ and the fact that $\mathbf{x}$ is a consistent family, it follows that $F \cap \mathcal{B}_i = [\langle i, x_i\rangle]^{\uparrow} \cap \mathcal{B}_i$, so that it is principal in $\mathcal{B}_i$.

Conversely, if F is closed for every $\mathcal{B}_i$, define x_i as the least element of $F \cap \mathcal{B}_i$. Obviously, $F = \bigcup_i [\langle i, x_i\rangle]^{\uparrow}$, so what remains to be show is that $\{x_i\}$ is a consistent family. But $f_{i|j}(x_j)$ is the projection of x_j on $\mathcal{B}_i$, so that $x_i \leq f_{i|j}(x_j)$ as $F \cap \mathcal{B}_i = x_i{}^{\uparrow}$. □

For an orthoposet $\mathcal{P}$ and a b.r.s. $\mathcal{S} \in \mathrm{BRS}(\mathcal{P})$, let $\wp^{\uparrow}(\mathcal{P}, \mathcal{S})$ denote the set of filters of $\mathcal{P}$ having a consistent representation in $\mathcal{S}$. This set can be given the structure of a poset by considering reverse inclusion of filters.

Proposition 4.4. *Given an orthoposet $\mathcal{P}$ and a b.r.s. $\mathcal{S} \in \mathrm{BRS}(\mathcal{P})$, for all $x \in \mathcal{P}$, one has $x^{\uparrow} \in \wp^{\uparrow}(\mathcal{P}, \mathcal{S})$ where $x^{\uparrow} = \{y \in \mathcal{P} | x \leq y\}$.*

Proof: This follows directly from proposition 4.1. □

Proposition 4.5. *For every filter F in $\wp^{\uparrow}(\mathcal{P}, \mathcal{S})$ with $\mathcal{S} \in \mathrm{BRS}(\mathcal{P})$, $F \neq \emptyset$ and either $\bot \in F$ or $\forall x \in \mathcal{P}, x \in F \Rightarrow x^{\bot} \notin F$.*

Proof: Let $\mathbf{x} = \{x_i\}$ be a consistent representation of F in $\mathcal{S}$. First, F is not empty since $\forall i, x_i \in F$. Now, suppose that $\bot \notin F$ and let y be an element of F. There is an indice i such that $y \in \mathcal{B}_i$, so that $x_i \leq y$. Now, since $\bot \notin F$, then $x_i \neq \bot$ so that $x_i \not\leq y^{\bot}$. As a consequence, since $y^{\bot}$ is also in $\mathcal{B}_i$, $y^{\bot} \notin F$. □

Proposition 4.6. *For all F in $\wp(\mathcal{P})$, F is in $\wp(\mathcal{P}, \mathbf{2}^{\mathcal{P}})$ if and only if $F \neq \emptyset$ and if $\bot \notin F$, then $\forall x \in F, x^{\bot} \notin F$.*

Proof: The $\Rightarrow$-direction corresponds to proposition 4.5. Let us prove the other direction, and for $y \in \mathcal{P} \setminus \{\top, \bot\}$, consider the intersection $F \cap \mathcal{B}_y$ where $\mathcal{B}_y$ is the boolean algebra generated by $\{y\}$. First, one has $\top \in F \cap \mathcal{B}_y$. Now, if $\bot \notin F$ then it cannot be that both y and $y^{\bot}$ are in $F \cap \mathcal{B}_y$. This shows that F is closed for $\mathcal{B}_y$. One concludes using 4.3. □

A direct consequence of proposition 4.4 is that given an orthoposet $\mathcal{P}$ and a b.r.s. $\mathcal{S} \in \mathrm{BRS}(\mathcal{P})$, $x \mapsto x^{\uparrow}$ is a poset embedding of $\mathcal{P}$ into $\wp^{\uparrow}(\mathcal{P}, \mathcal{S})$: it is one-to-one, and if $x \leq y$, then $y^{\uparrow} \subseteq x^{\uparrow}$. Thus, $\wp^{\uparrow}(\mathcal{P}, \mathcal{S})$ can be seen as a completion of the initial orthoposet $\mathcal{P}$. Let us define Compl($\mathcal{P}$) as the set $\{\wp^{\uparrow}(\mathcal{P}, \mathcal{S}) | \mathcal{S} \in \mathrm{BRS}(\mathcal{P})\}$. It can be given the structure of a lattice, using inclusion as partial order relation.

Proposition 4.7. Compl($\mathcal{P}$) *is complete lattice which greatest element is* $\wp^{\uparrow}(\mathcal{P}, \mathbf{2}^{\mathcal{P}})$ *and which join operation is defined by:*

$$\forall \mathbf{C} \subseteq \mathrm{Compl}(\mathcal{P}), \bigwedge \mathbf{C} = \{F | \forall \mathcal{C} \in \mathbf{C}, F \in \mathcal{C}\} \tag{15}$$

Proof: The fact that $\wp^{\uparrow}(\mathcal{P}, \mathbf{2}^{\mathcal{P}})$ is the greatest element of Compl($\mathcal{P}$) follows directly from propositions 4.5 and 4.6.

Now, let $\mathbf{C}$ be a subset of Compl($\mathcal{P}$), and for every $\mathcal{C} \in \mathbf{C}$, let $\mathcal{S}_{\mathcal{C}}$ be a b.r.s. in BRS($\mathcal{P}$) such that $\mathcal{C} = \wp^{\uparrow}(\mathcal{P}, \mathcal{S}_{\mathcal{C}})$. In terms of projective boolean subalgebras of $\mathcal{P}$, F is in $\mathcal{C}$ if and only if it is closed for every boolean subalgebra $\mathcal{B}$ in $\mathcal{S}_{\mathcal{C}}$. From proposition 3.12, one can define a b.r.s. $\mathcal{S}_{\wedge}$ using the union w.r.t. $\mathbf{C}$ of the projective boolean subalgebras of $\mathcal{S}_{\mathcal{C}}$. This definition directly implies that $\bigwedge \mathbf{C} = \wp^{\uparrow}(\mathcal{P}, \mathcal{S}_{\wedge})$. □

An open question which follows from this proposition is the characterization of the least element of Compl($\mathcal{P}$), since its existence is proved.

Let us now consider another problem related to the completions of an orthoposet. In general, only the partial order relation is preserved by the embedding $x \mapsto x^{\uparrow}$. Thus, even though one starts with an orthoposet, the orthocomplement operation may not have a counterpart in the completion.

In the next section, we will consider a property on completions of a given orthoposet $\mathcal{P}$ which permits to define a form of orthocomplementation in the completion, and will use this to construct an orthoposet embedding of $\mathcal{P}$ into a Heyting algebra.

5. ORTHOCOMPLETIONS

Given an orthoposet $\mathcal{P}$ and a completion $\mathcal{C}$ of it, proposition 4.5 has the following direct consequence:

$$\forall F \in \mathcal{C}, \forall x \in \mathcal{P}, x \in F \Rightarrow \left(\forall G \leq_{\mathcal{C}} F, x^{\perp} \in G \Rightarrow \perp \in G\right) \tag{16}$$

We use its converse to characterize a subset of Compl($\mathcal{P}$).

Definition 5.1 (Orthocompletion). Given a completion $\mathcal{C} \in$ Compl($\mathcal{P}$), $\mathcal{C}$ is an *orthocompletion* of $\mathcal{P}$ if and only if:

$$\forall F \in \mathcal{C}, \forall x \in \mathcal{P} \backslash F, \nexists G \leq_{\mathcal{C}} F, x^{\perp} \in G \, and \, x \notin G \tag{17}$$

Proposition 5.1. *Given an orthoposet* $\mathcal{P}$, $\wp^{\uparrow}(\mathcal{P}, \mathbf{2}^{\mathcal{P}})$ *is an orthocompletion of* $\mathcal{P}$.

Proof: As shown in proposition 4.6, a filter F is in $\wp^{\uparrow}(\mathcal{P}, \mathbf{2}^{\mathcal{P}})$ if and only if it is nonempty, and either $\bot \in F$ or $\forall x \in F, x^{\bot} \notin F$. So, let F be such a filter and let x be in $\mathcal{P} \setminus F$. Define filter G as $F \cup x^{\bot\uparrow}$. G is such that it contains $F \cup \{x^{\bot}\}$ but not x, so that it only remains to show that $G \in \wp^{\uparrow}(\mathcal{P}, \mathbf{2}^{\mathcal{P}})$. For this, suppose that there is an element $y \in \mathcal{P}$ such that $\{y, y^{\bot}\} \in G$. Since both F and $x^{\uparrow}$ are in $\wp^{\uparrow}(\mathcal{P}, \mathbf{2}^{\mathcal{P}})$, the only possibility, without loss of generality, is that $y \in F$ and $y^{\bot} \in x^{\bot\uparrow}$. But in that case, $x^{\bot} \leq y^{\bot}$ so that $y \leq x$ and finally $x \in F$, which is false. □

Equivalently, the property characterizing orthocompletions can be written as:

$$\forall F \in \mathcal{C}, \forall x \in \mathcal{P}, (\forall G \in \mathcal{C}, F \cup \{x\} \subseteq G \Rightarrow \bot \in G) \Rightarrow x^{\bot} \in F \tag{18}$$

Finally, this proposition can be written in a interesting third way. For this, dually to filters, define an *ideal* of a poset $\mathcal{P}$ as a subset $I \subseteq \mathcal{P}$ such that $\forall x \in I, \forall y \in \mathcal{P}, y \leq x \Rightarrow y \in I$. Let $\wp^{\downarrow}(\mathcal{P})$ denote the set of ideals of $\mathcal{P}$, and for $x \in \mathcal{P}$, define $x^{\downarrow}$ as $\{y \in \mathcal{P} | y \leq x\}$. Then, considering a completion $\mathcal{C}$ as a poset, for $x \in \mathcal{P}$, $x^{\uparrow\downarrow} = \{F \in \mathcal{C} | x \in F\}$ so that equation 17 can be expressed as:

$$\forall x \in \mathcal{P}, \left\{F \in \mathcal{C} | \forall G \leq_{\mathcal{C}} F, G \in x^{\uparrow\downarrow} \Rightarrow G \in \bot^{\uparrow\downarrow}\right\} = x^{\bot\uparrow\downarrow} \tag{19}$$

One can recognize in this expression the pseudocomplement operation of a Heyting algebra of the form $\wp^{\downarrow}(\mathcal{P})$ for any poset $\mathcal{P}$.

Heyting algebras (Birkhoff, 1967; Goldblatt, 1979) are a generalization of boolean algebras and are the models of intuitionistic logic. In particular, a Heyting algebra $\mathcal{H}$ is a distributive lattice with a least element $\bot$ and a pseudo-complement operation $\cdot \rightarrow \cdot$ such that:

$$\forall x, y, z \in \mathcal{H}, x \wedge y \leq z \Leftrightarrow x \leq y \rightarrow z \tag{20}$$

This permits to define a negation $\neg x$ as $x \rightarrow \bot$.

In the case of a Heyting algebra defined as the set of ideals of a poset, the meet and join operations correspond to the union and the intersection respectively, and the pseudo-complement is defined as:

$$I_1 \rightarrow I_2 = \{x | \forall x' \leq x, x' \in I_1 \Rightarrow x' \in I_2\} \tag{21}$$

If $\mathcal{P}$ is a poset with a least element $\bot$, one can instead consider a Heyting algebra formed of the nonempty ideals of $\mathcal{P}$, in which case the least element is $\{\bot\}$. The next proposition shows an orthocompletion $\mathcal{C}$ of a bounded orthoposet $\mathcal{P}$ permits to construct a Heyting algebra in which $\mathcal{P}$ can be embedded with preservation of the orthocomplementation.

Proposition 5.2. *Given an orthoposet $\mathcal{P}$ and an orthocompletion $\mathcal{C}$ of $\mathcal{P}$, the function $x \mapsto x^{\uparrow\downarrow}$ from $\mathcal{P}$ to $\wp^{\downarrow}(\mathcal{C})\backslash\{\emptyset\}$ is an orthoposet embedding, that is it is* *one-to-one, monotonous, and it verifies:*

$$\forall x \in \mathcal{P}, \neg(x^{\uparrow\downarrow}) = x^{\uparrow\downarrow} \rightarrow \perp^{\uparrow\downarrow} = (x^{\perp^{\uparrow\downarrow}}) \tag{22}$$

Proof: The injectivity and monotony of this function follow directly from its definition. The equality $\neg(x^{\uparrow\downarrow}) = (x^{\perp})^{\uparrow\downarrow}$ is equivalent to equation 19. □

This construction can be related to the different attempts that exist to embed quantum structures such as orthomodular lattices, orthoalgebras into classical logical structures, such as boolean algebras which are the usual models of classical logic. Embeddings into boolean algebras are presented in (Svozil, 1998; Calude *et al.*, 1999), where in particular it is shown than any orthoposet can be embedded into a Boolean algebra, with preservation of the partial order relation and the orthocomplementation (from results by Katrnoška, 1982 and by Zierler and Schlessinger, 1965). It is also shown that it is not possible to preserve other usual operations (such as join and meet) without drastic restrictions.

With regards to these results, the embedding which we have presented appears to be a weaker result, since boolean algebras are special kinds of Heyting algebras. However, considering Heyting algebras instead of boolean algebras do not permit to hope for stronger embeddings. The reason for this is that an element x of the original poset is mapped to $x^{\uparrow\downarrow}$ which verifies $x^{\uparrow\downarrow} = \neg\neg(x^{\uparrow\downarrow})$ (this follows from a double application of proposition 5.2) and that in a Heyting algebra $\mathcal{H}$, elements of the form $\neg x$ do form a boolean algebra. However, where are usually more orthocompletions of $\mathcal{P}$ than just $\mathbf{2}^{\mathcal{P}}$, as illustrated by the following proposition.

Proposition 5.3. *Given an orthoposet $\mathcal{P}$ and a projective boolean subalgebra $\mathcal{B}$ of $\mathcal{P}$, two elements $x \leq y$ of $\mathcal{P}$, there exists an orthocompletion $\mathcal{C}$ of $\mathcal{P}$ such that every filter in $\mathcal{C}$ is closed for $\mathcal{B}$.*

Proof: Using proposition 3.13, $\mathcal{B}$ can be turned into a b.r.s. $\mathcal{S}_{\mathcal{B}}$ by adding boolean subalgebras $\mathcal{B}_z = \{\top, \perp, z, z^{\perp}\}$ for $z \in \mathcal{P} \setminus \mathcal{B}$. Due to the presence of $\mathcal{B}$, filters in $\mathcal{C}_{\mathcal{B}} = \wp^{\uparrow}(\mathcal{P}, \mathcal{S}_{\mathcal{B}})$ are obviously closed for $\mathcal{B}$. Thus, what remains to show is that $\mathcal{C}_{\mathcal{B}}$ is an orthocompletion of $\mathcal{P}$.

Let x be an element of $\mathcal{P}$ and F be a filter in this completion such that $z \notin F$. Let $f_{\mathcal{B}}$ be the least element element of $F \cap \mathcal{B}$ and for $y \in \mathcal{P}$, let $\rho_{\mathcal{B}}(y)$ denote its the projection on $\mathcal{B}$, and define $G = F \cup x^{\perp\uparrow} \cup (f_{\mathcal{B}} \wedge \rho_{\mathcal{B}}(x^{\perp}))^{\uparrow}$. We first show that G is in $\mathcal{C}_{\mathcal{B}}$. For this, first remark that G is closed for $\mathcal{B}$. Now, let y be in $\mathcal{P} \setminus \mathcal{B}$. We show that G is also closed for $\mathcal{B}_y$. Suppose that it is not the case, so that $G \cap \mathcal{B}_y = \{\top, y, y^{\perp}\}$. Without loss of generality, six cases are to be considered.

1. The first three possibilities, which are $\{y, y^{\perp}\} \subseteq F$, $\{y, y^{\perp}\} \subseteq x^{\perp\uparrow}$ and $\{y, y^{\perp}\} \subseteq (f_{\mathcal{B}} \wedge \rho_{\mathcal{B}}(x^{\perp}))^{\uparrow}$ are obviously impossible.

2. If $y \in F$ and $x^{\perp} \leq y^{\perp}$, then $y \leq x$, which implies $x \in F$ so that it is impossible.
3. If $y \in F$ and $f_{\mathcal{B}} \wedge \rho_{\mathcal{B}}(x^{\perp}) \leq y^{\perp}$, then one has $f_{\mathcal{B}} \leq \rho_{\mathcal{B}}(y)$ and $f_{\mathcal{B}} \wedge \rho_{\mathcal{B}}(x^{\perp}) \leq \rho_{\mathcal{B}}(y)^{\perp}$. As a consequence, $f_{\mathcal{B}} \wedge \rho_{\mathcal{B}}(x^{\perp}) = \perp$ and $G \cap \mathcal{B}_y \neq \{\top, y, y^{\perp}\}$ since it also contains $\perp$. Thus, it is impossible.
4. The last case is when $x^{\perp} \leq y$ and $f_{\mathcal{B}} \wedge \rho_{\mathcal{B}}(x^{\perp}) \leq y^{\perp}$. Then, once again, this implies that $f_{\mathcal{B}} \wedge \rho_{\mathcal{B}}(x^{\perp}) = \perp$ which is not possible either.

We have thus shown that G is closed for every $\mathcal{B}_y$ in $\mathcal{C}_{\mathcal{B}}$, so that G is in $\mathcal{C}_{\mathcal{B}}$. What remains is to ensure that $x \notin G$. But $x \in G$ would imply $f_{\mathcal{B}} \wedge \rho_{\mathcal{B}}(x^{\perp}) \leq x$ and then $f_{\mathcal{B}} \wedge \rho_{\mathcal{B}}(x^{\perp}) \leq \rho_{\mathcal{B}}(x^{\perp})^{\perp}$. As a consequence, one would have $f_{\mathcal{B}} \wedge \rho_{\mathcal{B}}(x^{\perp}) = \perp$ and $f_{\mathcal{B}} \leq \rho_{\mathcal{B}}(x^{\perp})^{\perp} \leq x$. This finally would imply that $x \in F$, which is false. □

This shows that for every projective boolean subalgebra $\mathcal{B}$ of $\mathcal{P}$, it is possible to embed $\mathcal{P}$ in a Heyting algebra in such a way that for all x and y in $\mathcal{B}$, one has $x^{\uparrow\downarrow} \rightarrow y^{\uparrow\downarrow} = (x^{\perp} \vee y)^{\uparrow\downarrow}$ and $x^{\downarrow\uparrow} \wedge y^{\uparrow\downarrow} = (x \wedge y)^{\uparrow\downarrow}$.

Thus, the use of orthocompletions of $\mathcal{P}$ permits to construct Heyting algebras in which $\mathcal{P}$ can be embedded. Moreover, this suggest that Heyting algebras and intuitionistic logic are natural candidates into which to embed quantum structures and can be related to some existing studies on the relationship between orthomodular structures and Heyting algebras in the one hand, and quantum logic and intuitionistic logic in the other hand (Coecke *et al.*, 2000; Coecke, 2002). This can be justified as follows:

We have shown a way to decompose an orthoposet $\mathcal{P}$ as a collection of boolean algebras using a boolean representation system in BRS($\mathcal{P}$). Furthermore, these boolean representation systems permit to define completions of $\mathcal{P}$, which elements correspond to partial descriptions of the studied system. Elements of $\mathcal{P}$ are then special partial descriptions, those which can be "observed." Now, a simple way to study this system logically would be to associate to each logical proposition p a set I_p of partial descriptions, those which provide enough information to ensure that p actually holds. In this way, it appears that the set I_p has to be an ideal of the completion (since adding information preserves the provability of p), so that considering the set of ideals of the completion, which is a Heyting algebra, follows naturally.

6. CONCLUSION AND PERSPECTIVES

We have presented representation systems, which can be seen as a formalization of the notion of partial description of a system from different points of view. We have then shown that with the restriction that each point of view corresponds to a boolean algebra (thus behaving in a classical way), one could characterize the class of orthoposets and use boolean representation systems to provide a decomposition of orthoposets into boolean algebras.

Using these decomposition, we have introduced a general way to construct completions of an orthoposet. These completions permit to consider more partial descriptions, with the restriction that the added ones may not in general be the result of some observation process (contrary to the elements of the original orthoposet).

Finally, we have shown that some (at least one) of these completions could be used to define a Heyting algebra in which the original orthoposet can be embedded while preserving the partial order relation and the orthocomplementation operation.

These constructions suggest a general way to provide a logical study of quantum systems. In our approach, embedding quantum structures into Heyting algebras permits to focus on partial descriptions, and more precisely, on those partial description given by the original quantum structure, which are interpreted as results of observation processes. Finally, it can be noted that this interpretation of an orthoposet and the resulting embedding into a Heyting algebra also permits to simply extend this study by adding a dynamic dimension, as initiated in (Brunet and Jorrand, 2003, in press).

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